CHAPTER I

THE ABSTRACT NATURE OF MATHEMATICS

THE STUDY of mathematics is apt to commence in disappointment. The important applications of the science, the theoretical interest of its ideas, and the logical rigour of its methods, all generate the expectation of a speedy introduction to processes of interest. We are told that by its aid the stars are weighed and the billions of molecules in a drop of water are counted. Yet, like the ghost of Hamlet's father, this great science eludes the efforts of our mental weapons to grasp it—''Tis here, 'tis there, 'tis gone'—and what we do see does not suggest the same excuse for illusiveness as sufficed for the ghost, that it is too noble for our gross methods. 'A show of violence,' if ever excusable, may surely be 'offered' to the trivial results which occupy the pages of some elementary mathematical treatises.

The reason for this failure of the science to live up to its reputation is that its fundamental ideas are not explained to the student disentangled from the technical procedure which has been invented to facilitate their exact presentation in particular instances. Accordingly, the unfortunate learner finds himself struggling to acquire a knowledge of a mass of details which are not illuminated by any general conception. Without a doubt, technical facility is a first requisite for valuable mental activity: we shall fail to appreciate the rhythm of Milton, or the passion of Shelley, so long as we find it necessary to spell the words and are not quite certain of the forms of the individual letters. In this sense there is no royal road to learning. But it is equally an error to confine attention to technical processes, excluding consideration of general ideas. Here lies the road to pedantry.

The object of the following chapters is not to teach mathematics, but to enable students from the very beginning of their course to know what the science is about, and why it is necessarily the foundation of exact thought as applied to natural phenomena. All allusion in what follows to detailed deductions in any part of the science will be inserted merely for the purpose of example, and care will be taken to make the general argument comprehensible, even if here and there some technical process or symbol which the reader does not understand is cited for the purpose of illustration.

The first acquaintance which most people have with mathematics is through arithmetic. That two and two make four is usually taken as the type of a simple mathematical proposition which everyone will have heard of. Arithmetic, therefore, will be a good subject to consider in order to discover, if possible, the most obvious characteristic of the science. Now, the first noticeable fact about arithmetic is that it applies to everything, to tastes and to sounds, to apples and to angels, to the ideas of the mind and to the bones of the body. The nature of the things is perfectly indifferent, of all things it is true that two and two make four. Thus we write down as the leading characteristic of mathematics that it deals with properties and ideas which are applicable to things just because they are things, and apart from any particular feelings, or emotions, or sensations, in any way

connected with them. This is what is meant by calling mathematics an abstract science.

The result which we have reached deserves attention. It is natural to think that an abstract science cannot be of much importance in the affairs of human life, because it has omitted from its consideration everything of real interest. It will be remembered that Swift, in his description of Gulliver's voyage to Laputa, is of two minds on this point. He describes the mathematicians of that country as silly and useless dreamers, whose attention has to be awakened by flappers. Also, the mathematical tailor measures his height by a quadrant, and deduces his other dimensions by a rule and compasses, producing a suit of very ill-fitting clothes. On the other hand, the mathematicians of Laputa, by their marvellous invention of the magnetic island floating in the air, ruled the country and maintained their ascendancy over their subjects. Swift, indeed, lived at a time peculiarly unsuited for gibes at contemporary mathematicians. Newton's *Principia* had just been written, one of the great forces which have transformed the modern world. Swift might just as well have laughed at an earthquake.

But a mere list of the achievements of mathematics is an unsatisfactory way of arriving at an idea of its importance. It is worth while to spend a little thought in getting at the root reason why mathematics, because of its very abstractness, must always remain one of the most important topics for thought. Let us try to make clear to ourselves why explanations of the order of events necessarily tend to become mathematical.

Consider how all events are interconnected. When we see the lightning, we listen for the thunder; when we hear the wind, we look for the waves on the sea; in the chill autumn, the leaves fall. Everywhere order reigns, so that when some circumstances have been noted we can foresee that others will also be present. The progress of science consists in observing these interconnexions and in showing with a patient ingenuity that the events of this ever-shifting world are but examples of a few general connexions or relations called laws. To see what is general in what is particular and what is permanent in what is transitory is the aim of scientific thought. In the eye of science, the fall of an apple, the motion of a planet round a sun, and the clinging of the atmosphere to the earth are all seen as examples of the law of gravity. This possibility of disentangling the most complex evanescent circumstances into various examples of permanent laws is the controlling idea of modern thought.

Now let us think of the sort of laws which we want in order completely to realize this scientific ideal. Our knowledge of the particular facts of the world around us is gained from our sensations. We see, and hear, and taste, and smell, and feel hot and cold, and push, and rub, and ache, and tingle. These are just our own personal sensations: my toothache cannot be your toothache, and my sight cannot be your sight. But we ascribe the origin of these sensations to relations between the things which form the external world. Thus the dentist extracts not the toothache but the tooth. And not only so, we also endeavour to imagine the world as one connected set of things which underlies all the perceptions of all people. There is not one world of things for my sensations and another for yours, but one world in which we both exist. It is the same tooth both for dentist and patient. Also we hear and we touch the same world as we see.

It is easy, therefore, to understand that we want to describe the connexions between these external things in some way which does not depend on any particular sensations, nor even on all the sensations of any particular person. The laws satisfied by the course of events in the world of external things are to be described, if possible, in a neutral universal fashion, the same for blind men as for deaf men, and the same for beings with faculties beyond our ken as for normal human beings.

But when we have put aside our immediate sensations, the most serviceable part—from its clearness, definiteness, and universality—of what is left is composed of our general ideas of the abstract formal properties of things; in fact, the abstract mathematical ideas mentioned above. Thus it comes about that, step by step, and not realizing the full meaning of the process, mankind has been led to search for a mathematical description of the properties of the universe, because in this way only can a general idea of the course of events be formed, freed from reference to particular persons or to particular types of sensation. For example, it might be asked at dinner: 'What was it which underlay my sensation of sight, yours of touch, and his of taste and smell?' the answer being 'an apple'. But in its final analysis, science seeks to describe an apple in terms of the positions and motions of molecules, a description which ignores me and you and him, and also ignores sight and touch and taste and smell. Thus mathematical ideas, because they are abstract, supply just what is wanted for a scientific description of the course of events.

This point has usually been misunderstood, from being thought of in too narrow a way. Pythagoras had a glimpse of it when he proclaimed that number was the source of all things. In modern times the belief that the ultimate explanation of all things was to be found in Newtonian mechanics was an adumbration of the truth that all science as it grows towards perfection becomes mathematical in its ideas.

CHAPTER 2

VARIABLES

MATHEMATICS AS a science commenced when first someone, probably a Greek, proved propositions about *any* things or about *some* things, without specification of definite particular things. These propositions were first enunciated by the Greeks for geometry; and, accordingly, geometry was the great Greek mathematical science. After the rise of geometry centuries passed away before algebra made a really effective start, despite some faint anticipations by the later Greek mathematicians.

The ideas of *any* and of *some* are introduced into algebra by the use of letters, instead of the definite numbers of arithmetic. Thus, instead of saying that $2+3=3+2$, in algebra we generalize and say that, if x and y stand for *any* two numbers, then $x+y=y+x$. Again, in the place of saying that $3>2$, we generalize and say that if x be *any* number there exists *some* number (or numbers) y such that $y>x$. We may remark in passing that this latter assumption—for when put in its strict ultimate form it is an assumption—is of vital importance, both to philosophy and to mathematics; for by it the notion of infinity is introduced. Perhaps it required the introduction of the arabic numerals, by which the use of letters as standing for definite numbers has been completely discarded in mathematics, in order to suggest to mathematicians the technical convenience of the use of letters for the ideas of *any* number and *some* number. The Romans would have stated the number of the year